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Nonlinear propagation of electromagnetic pulses in two-level media under strong coupling

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Abstract. We consider the problem of the propagation of short-duration light pulses in a two-level medium in the regime of strong interaction, near the resonance, with no restriction on the strength of the coupling between field and medium. We prove that the local variations of the population difference created by the electric field actually induce a nonlinear effect by coupling the fundamental of the field Fourier component to its harmonics. As a result, the multiscale averaging limit of the Maxwell–Bloch system is a new system of coupled nonlinear Schrödinger equations for the slowly varying envelope of the electric field. This system has one integrable limit when the electric field is *circularly polarized*: it is the scalar nonlinear Schrödinger equation. Moreover, it possesses three constants of the motion and a Hamiltonian formulation. Then we prove that our system is non-integrable as it does not pass the Painlevé test. Finally, a stability analysis shows that it is *modulationally unstable* and thus it propagates steady pulses of coherent light.

1. Introduction

We report here results of a theoretical study of the nonlinear propagation of an electromagnetic field in a dielectric-material medium. This is an old and wide-ranging problem [1] which we restrict to the case of the interaction between classical radiation and a two-level quantized medium. The basic model equations are the Maxwell equation for the electromagnetic field and the Schrödinger (or Bloch) equation for the dipole moment operator. This semi-classical approach is known to be quite effective in the absence of spontaneous emission processes. For the general problem of the interaction of electromagnetic radiation with a molecular fluid, we refer to [2] (and the references quoted therein) where a complete treatment of the linear problem is given.

The problem which we are interested in is that of the modulation of the electric-field envelope resulting from the transient response of the medium in the regime of strong interaction. Here, strong interaction means that the *nonlinear coupling* between field and medium is a dominating process. This can be achieved in two ways: either by launching large-amplitude laser pulses or by considering a medium with a strong coupling. The first situation is described in the work of McCall and Hahn [3] where the coupling is weak with respect to the field intensity. We consider here the second case when the magnitude of the coupling is not a *small parameter* of the model.

Next the exciting pulse is considered to be of short duration compared with the relaxation times of the medium and hence we neglect the relaxation effects. This has proved to be a relevant approximation for laser pulses of duration of order 10 to 100 times the relaxation time T_2, as the experiments do confirm the lossless pulse propagation [4].

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Since the works of McCall and Hahn [3], we know that a short-duration strong laser pulse launched in a low-density ($< 10^{18}$ part cm^3) dielectric medium at a frequency close to the resonant frequency propagates with wave number $k = \omega c$ (c is the light velocity in the medium), after having experienced an envelope reshaping to a soliton form. Many efforts have been made since then to extend and improve the theoretical model for self-induced transparency (SIT).

A first instance of a more accurate description of the model has been proposed in [6] where the only approximation made on the general equation is that the interaction process does not produce backscattering of electromagnetic waves. The resulting model equation, called the reduced Maxwell–Bloch system, has then been proved to be integrable in [7] and hence to support a an N-soliton solution (see also [8]). It is worth remarking that the reduced Maxwell–Bloch system reduces to the SIT equations through a *slowly varying envelope approximation* (SVEA) limiting procedure, and hence that the integrability of the SIT system has actually been proved almost simultaneously in [7] and [9].

A different approximation method has been used in [10] to include the effects of the second-order derivatives and to take into account the relative dimension of the Rabi frequency of the pulse versus the atomic transition frequency. This approach has allowed for treating much shorter pulses than the McCall and Hahn approach. Another example is found in [11] where the preceding approach relaxing the SVEA is extended in a systematic way by means of *asymptotics beyond all orders*. As a result, it is shown that the pulse velocities can only take discrete values, contrary to what is expected from a soliton which propagates at arbitrary velocities. Last we mention the work [12] where the intrinsic nonlinearity of the polarization (nonlinear refractive index) and the group-velocity dipersion have been added to the coupling nonlinearity. The resulting system is still integrable [13].

In all the studies based upon SVEA, the magnitude of the coupling between fields and dipole population is the basic small parameter used for the multiscale expansion on the Maxwell–Bloch system. We want to consider here the general case when no particular restrictions are imposed on the intensity of the coupling. To make this study, we propose a rigorous asymptotic analysis of the basic system and consider, in particular, all the harmonic Fourier components of the fields (infinite series expansion) and assume that the envelopes vary slowly in space and time with respect to the phases.

The resulting model equation is a system of coupled nonlinear Schrödinger equations, the coupling occurring between the two transverse components of the electromagnetic field. In this system the magnitude of the nonlinearity equals the value of the coupling parameter of the original starting Maxwell–Bloch system.

To our knowledge this system is new and we prove that it is a Hamiltonian system which possesses three constants of motion. By means of a Painlevé analysis we prove that the system is in general non-integrable, except in the case of circularly polarized waves, when it reduces to the scalar nonlinear Schrödinger equation. Our model is then shown to be *modulationally unstable (Benjamin–Feir instability), and hence the nonlinearity is a* natural source for solitary wave generation. The one-soliton solution is a two-parameter family of localized envelope solutions which corresponds to a circularly polarized electric field.

2. Basic equation and dispersion relation

Applying the usual procedure of molecular averaging over the dipoles, the material medium is completely described by the polarization vector $\boldsymbol{P}$ and the population difference N, see e.g. [14]. As a result the basic equations of our study are in the isotropic case (dilute gas or when the fields are polarized along principal axes of a crystal [14, 15])

$$
\begin{aligned}
&\frac{\partial^2}{\partial t^2}\boldsymbol{P} + \Omega^2 \boldsymbol{P} = -\kappa N \boldsymbol{E} \\
&\frac{\partial}{\partial t} N = \frac{2}{\hbar\Omega} \boldsymbol{E} \cdot \frac{\partial}{\partial t} \boldsymbol{P} \\
&\nabla \wedge \nabla \wedge \boldsymbol{E} + \frac{\eta^2}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{E} = -\mu_0 \frac{\partial^2}{\partial t^2}\boldsymbol{P} .
\end{aligned}
\tag{2.1}
$$

In the above system, Ω is the dipole transition frequency, $\eta^2 = \epsilon/\epsilon_0$ the refractive index and

$$
\kappa = \frac{2\Omega}{\hbar}\left(\frac{\eta^2+2}{3}\right)^2 \frac{1}{3}\langle|\mu_{12}|^2\rangle \tag{2.2}
$$

is a constant characterizing the dielectric medium (μ_{12} is the electric dipole moment and the average is taken over the orientations). The polarization $\boldsymbol{P}$ is actually the *source* term and $\boldsymbol{E}$ the macroscopic field, which explains the presence in (2.1) of the Lorentz local-field correction factor $(\eta^2+2)/3$.

First of all we reduce the system (2.1) by considering the propagation in only one direction, say z. Then it is quite useful to rescale the variables and normalize the fields by defining

$$
T = \Omega t \qquad Z = \frac{\eta}{c}\Omega z \tag{2.3}
$$

$$
\mathcal{N}(Z,T) = N(z,t)N_0^{-1} \tag{2.4}
$$

$$
\mathcal{P}(Z,T) = \mu_0 \frac{c}{\eta}\boldsymbol{P}(z,t)\sqrt{\frac{2}{N_0\hbar\Omega\mu_0}} \tag{2.5}
$$

$$
\mathcal{E}(Z,T) = \frac{\eta}{c}\boldsymbol{E}(z,t)\sqrt{\frac{2}{N_0\hbar\Omega\mu_0}} \tag{2.6}
$$

where N_0 is the number of active dipoles per unit volume. The sytem (2.1) becomes the following *dimensionless* system:

$$
\partial_T^2\mathcal{P} + \mathcal{P} = -\alpha\mathcal{N}\mathcal{E} \qquad \partial_T\mathcal{N} = \mathcal{E}\cdot\partial_T\mathcal{P} \qquad \left(\partial_T^2 - J_z\partial_Z^2\right)\mathcal{E} = -\partial_T^2\mathcal{P} \tag{2.7}
$$

where we have defined

$$
J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{2.8}
$$

Note that the above structure follows from the starting equation (2.1) when $\boldsymbol{E}$ is assumed to depend on the variable Z only, which correspond to neglecting transverse modulational effects.

The equation (2.7) is our basic model and it is characterized by one coupling constant

$$
\alpha = \frac{N_0}{\hbar\Omega}\left(\frac{\eta^2+2}{3}\right)^2 \frac{1}{3}\langle|\mu_{12}|^2\rangle\mu_0\frac{c^2}{\eta^2} \tag{2.9}
$$

and by the normalization of the population inversion density

$$
\mathcal{N}(Z,T) \in [-1,1] \tag{2.10}
$$

($\mathcal{N} = -1$ corresponds to all atoms in the fundamental). Note that the constant $\alpha\mathcal{N}_0$, where $\mathcal{N}_0$ is the initial normalized population-density difference, is *dimensionless* and *irreducible* (it cannot be scaled off), and hence it is characteristic of the strengh of the interaction between field and medium. In this work we shall consider only the case of an attenuator, that is $\mathcal{N}_0 < 0$.

At this point it is worth remarking that the magnitude of the coupling constant α will have to be carefully considered with respect to the intensity of the electric field. Indeed, as we will make an asymptotic expansion in powers of a small parameter, we will need to compare this parameter on the one hand with the basic dimensional constant of our system, on the other hand with the magnitude of the applied electromagnetic field. In particular, for a dilute gas α is of order 10^{-9} while for a crystal its order is 10^{-2}.

We will make use in the following of the dispersion relation of the coupled system of partial differential equations (2.7) which is obtained by looking at the linear limit:

$$\mathcal{P}(Z,T) = \mathcal{P}_0 \exp[\mathrm{i}(\omega T - kZ)] + \mathrm{cc} \tag{2.11}$$

$$\mathcal{E}(Z,T) = \mathcal{E}_0 \exp[\mathrm{i}(\omega T - kZ)] + \mathrm{cc} \tag{2.12}$$

$$\mathcal{N}(Z,T) = \mathcal{N}_0 \,. \tag{2.13}$$

The non-vanishing solution of (2.7) holds for

$$(\omega^2 - 1)(\omega^2 - k^2) + \alpha\mathcal{N}_0\omega^2 = 0 \,. \tag{2.14}$$

The above equation furnishes the linear dispersion relation of the system (2.7).

3. Self-induced transparency in low-density media

In their original derivation [3], McCall and Hahn have started the slowly varying envelope approximation (SVEA) with a number of *a priori* assumptions. These asumptions will be *derived* here by seeking a solution of (2.7) under the most general following form:

$$\begin{aligned}
\mathcal{E}(Z,T) &= \sum_{j=0}^{\infty} \epsilon^j \sum_{n=-\infty}^{+\infty} E_j^n(\xi,\tau) \exp[\mathrm{i}n(\omega T - kZ)] \\
\mathcal{P}(Z,T) &= \sum_{j=0}^{\infty} \epsilon^j \sum_{n=-\infty}^{+\infty} P_j^n(\xi,\tau) \exp[\mathrm{i}n(\omega T - kZ)] \\
\mathcal{N}(Z,T) &= \sum_{j=0}^{\infty} \epsilon^j \sum_{n=-\infty}^{+\infty} N_j^n(\xi,\tau) \exp[\mathrm{i}n(\omega T - kZ)] \,.
\end{aligned} \tag{3.1}$$

The basic hypotheses here are

$$\alpha = \epsilon^2\delta \qquad 1 - \omega^2 = \mathcal{O}(\epsilon^2) \qquad \omega = k + \mathcal{O}(\epsilon^2) \,. \tag{3.2}$$

Hence the small parameter ϵ measures the intensity of the coupling through the definition of α hereabove: the present case corresponds for instance to a medium of *low density*. Then the other definitions in (3.2) have the following meaning: (i) the firing light pulse has a frequency equal (or very close to) the resonant frequency, and we shall be talking of *resonant pulse propagation*, (ii) the field propagates in the medium as in vacuum ($\omega \simeq k$). Moreover, the field is also supposed to be transverse inside the medium, that is

$$E_j^n(\xi,\tau)_z = P_j^n(\xi,\tau)_z = 0 \,. \tag{3.3}$$

These hypotheses completely determine the resulting limit equations: under the SVEA scaling

$$\partial_Z = \epsilon\partial_\xi \qquad \partial_T = \epsilon\partial_\tau \tag{3.4}$$

only non-trivial solutions will be obtained for a series of constraints on the different Fourier components in (3.1), which can be summarized by rewriting (3.1) as follows:

$$\begin{aligned}
\mathcal{E} &= \boldsymbol{E}_0^1(\xi,\tau)\,\mathrm{e}^{\mathrm{i}(\omega T - KZ)} + \mathrm{CC} + \mathcal{O}(\epsilon) \\
\mathcal{P} &= \epsilon \boldsymbol{P}_1^1(\xi,\tau)\,\mathrm{e}^{\mathrm{i}(\omega T - KZ)} + \mathrm{CC} + \mathcal{O}(\epsilon^2) \\
\mathcal{N} &= N_0^0(\xi,\tau) + \mathcal{O}(\epsilon)\,.
\end{aligned} \tag{3.5}$$

The above expressions actually constitute the starting assumptions in [3]. We do not rederive here the results but the interested reader will find the essence of the method in the next section for the derivation of our equation (details in the appendix), and this method works exactly in the same way for both cases.

Finally the system (2.7) becomes at first order in ϵ the sharp-line limit (no inhomogeneous broadening) of the SIT equations of McCall and Hahn [3]

$$\begin{aligned}
&2\mathrm{i}\omega\partial_\tau \boldsymbol{P}_1^1 = -\delta\, N_0^0 \boldsymbol{E}_0^1 \\
&\partial_\tau N_0^0 = \mathrm{i}\omega(\boldsymbol{P}_1^1 \cdot (\boldsymbol{E}_0^1)^* - (\boldsymbol{P}_1^1)^* \cdot \boldsymbol{E}_0^1) \\
&(\partial_\tau - \partial_\xi)\boldsymbol{E}_0^1 = \frac{\mathrm{i}}{2}\omega \boldsymbol{P}_1^1\,.
\end{aligned} \tag{3.6}$$

This system has been shown to be integrable by Lamb [9] and later in [16] to have the *mathematical* property of transparency: the continuous spectrum (or background radiation) is exponentially vanishing as the pulse propagates in the medium. Hence, any firing pulse is *reshaped* to a pure soliton structure which then propagates freely as it would do in a transparent medium.

4. Pulse propagation under strong coupling

Now no constraint is assumed on the size of the parameter α or on the relative values of ω and k. The same infinite series expansion as (3.1) is used, namely

$$\begin{aligned}
\mathcal{E}(Z,T) &= \sum_{j=0}^{\infty} \epsilon^j \sum_{n=-\infty}^{+\infty} E_j^n(\xi,\tau)\exp[\mathrm{i}n(\omega T - kZ)] \\
\mathcal{P}(Z,T) &= \sum_{j=0}^{\infty} \epsilon^j \sum_{n=-\infty}^{+\infty} P_j^n(\xi,\tau)\exp[\mathrm{i}n(\omega T - kZ)] \\
\mathcal{N}(Z,T) &= \sum_{j=0}^{\infty} \epsilon^j \sum_{n=-\infty}^{+\infty} N_j^n(\xi,\tau)\exp[\mathrm{i}n(\omega T - kZ)]
\end{aligned} \tag{4.1}$$

(for notational convenience, we forget from now on the bold face characters indicating that $\mathcal{E}$ and $\mathcal{P}$ are vectors of $\mathbb{R}^3$). The reality of the fields imposes, of course, that

$$E_j^{-n} = \bar{E}_j^n \qquad P_j^{-n} = \bar{P}_j^n \qquad N_j^{-n} = \bar{N}_j^n\,. \tag{4.2}$$

There is finally the natural condition that the fields $\mathcal{E}$ and $\mathcal{P}$ (but not $\mathcal{N}$) have no zero frequency mode:

$$E_j^0 = P_j^0 = 0 \qquad j = 0, 1, \ldots, \infty\,. \tag{4.3}$$

The amplitude components in (4.1) depend on the two *slow variables* ξ and τ. Looking for a travelling-wave solution, we set

$$\xi = \epsilon(Z - VT) \tag{4.4}$$

where V will be determined from the lowest orders in (2.7) and will have to be the group velocity of the wave (we will have to check that $V = \mathrm{d}\omega/\mathrm{d}k$). For the slow time τ, since the propagation at the group velocity is already included in (4.4), only small deviations at the next order are allowed, namely

$$\tau = \epsilon^2 T . \tag{4.5}$$

The technique then is only algebraic: it consists in examining the system (2.7) in which (4.1) is inserted at each successive order in ϵ up to extracting a *closed* evolution equation at the lowest non-trivial order. The reader will find the basic principles of this method of multiscale expansion in [17] and fundamental considerations about validity, consistency and universality in [18].

A first simple consequence of this approach is that we demonstrate that the first-order variations of the electric and polarization fields are circularly polarized, namely

$$E_{1z}^n = P_{1z}^n = 0 . \tag{4.6}$$

Then the system (2.7) leads to a closed system of evolution equations for the first-order two-component field $\boldsymbol{E}_1^1(\xi, \tau)$. The derivation of this system is reported in the appendix; we only want to point out here the following essential features of this derivation.

(i) When no ordering of the constant α is required, the order $n = 0$ forms a closed system if all its harmonics vanish, that is to say

$$P_0^n = E_0^n = 0 \qquad n = 1, 2, \ldots, \infty \tag{4.7}$$

and, as a consequence, $N_0^n = 0$. Hence the only non-vanishing quantity at order zero in ϵ is N_0^0, the initial normalized pupulation difference

$$N_0^0 \equiv \mathcal{N}_0 . \tag{4.8}$$

(ii) The order $n = 1$ furnishes on the one hand the dispersion relation (2.14), and on the other hand the result

$$E_{1x}^n = P_{1x}^n = E_{1y}^n = P_{1y}^n = 0 \qquad n > 1 \tag{4.9}$$

and

$$N_1^n = 0 \qquad n > 0 . \tag{4.10}$$

(iii) The order $n = 2$ furnishes on the one hand the velocity

$$V = \frac{k}{\omega} \frac{\omega^2(1 - \omega^2)}{k^2 - \omega^4} \equiv \frac{\mathrm{d}\omega}{\mathrm{d}k} \tag{4.11}$$

which is indeed the group velocity, on the other hand a system of coupled differential equations for the fields E_j^n, P_j^n and N_j^n for $j = 1, 2$ and $n = 1, 2$, and

$$N_1^0 = 0 . \tag{4.12}$$

(iv) To close the system obtained above, it is necessary to eliminate the second-order terms, which is done by looking at the expansion up to the third order in ϵ.

The final result is the following new system of coupled nonlinear Schrödinger equations for the vector field:

$$\boldsymbol{E}_1^1(\xi, \tau) = \begin{pmatrix} \psi(\xi, \tau) \\ \phi(\xi, \tau) \end{pmatrix} \tag{4.13}$$

$$\begin{aligned} &\mathrm{i}\lambda \partial_\tau \psi - \mu \partial_\xi^2 \psi - \alpha \left(a|\psi|^2 + |\phi|^2 \right) \psi = \alpha(a - 1)\phi^2 \bar{\psi} \\ &\mathrm{i}\lambda \partial_\tau \phi - \mu \partial_\xi^2 \phi - \alpha \left(a|\phi|^2 + |\psi|^2 \right) \phi = \alpha(a - 1)\psi^2 \bar{\phi} . \end{aligned} \tag{4.14}$$

Hereabove the characteristic constants λ, μ and a can be written by help of (2.14) as functions of only ω and k:

$$\lambda = \frac{2}{\omega}\frac{1-\omega^2}{1+\omega^2}\frac{k^2-\omega^4}{\omega^2-k^2} \qquad \mu = \omega^2\frac{(1-\omega^2)^2}{1+\omega^2}\frac{3k^2+\omega^4}{(\omega^4-k^2)^2} \qquad a = \frac{3+\omega^2}{2(1+\omega^2)}\,. \tag{4.15}$$

Note that λ can be scaled off in τ, μ in ξ and α in the amplitude of ψ and ϕ. Then a is the characteristic dimensionless constant of this system and it depends only on the input frequency ω.

The above system now serves as the basic tool to study the propagation of a pulse of transverse electromagnetic field in a two-level sytem. First of all, it is useful for the discussion to report hereafter a summary of the different relationships between the fields that are demonstrated in the appendix. Inserting in (4.1) all the restrictions on the Fourier components listed above together with those derived in the appendix, we have, in short, the expansion

$$\begin{aligned}
\mathcal{E}(Z,T) &= \epsilon E_1^1(\xi,\tau)\exp[\mathrm{i}(\omega T - kZ)] + \mathcal{O}(\epsilon^2) + \mathrm{cc} \\
\mathcal{P}(Z,T) &= \epsilon P_1^1(\xi,\tau)\exp[\mathrm{i}(\omega T - kZ)] + \mathcal{O}(\epsilon^2) + \mathrm{cc} \\
\mathcal{N}(Z,T) &= \mathcal{N}_0 + \epsilon^2\sum_{n=0}^{2} N_2^n(\xi,\tau)\exp[\mathrm{i}n(\omega T - kZ)] + \mathcal{O}(\epsilon^3) + \mathrm{cc}
\end{aligned} \tag{4.16}$$

with the relations

$$P_1^1 = -\frac{\alpha\mathcal{N}_0}{1-\omega^2}E_1^1 \qquad N_2^0 = -\frac{\alpha\mathcal{N}_0}{1-\omega^2}\frac{1+\omega^2}{1-\omega^2}E_1^1\bar{E}_1^1 \tag{4.17}$$

$$N_2^1 = 0 \qquad N_2^2 = \tfrac{1}{2}E_1^1P_1^1 \tag{4.18}$$

and the nonlinear equation (4.14) for the field E_1^1.

Comments.

(i) The coupling constant α, introduced in (2.7), is indeed fundamental as it measures now the strength of the nonlinearity. This nonlinearity is then a direct consequence of the coupling of the electric field $\mathcal{E}$ with the polarization field $\mathcal{P}$ through the population inversion $\mathcal{N}$.

(ii) Considering (4.16), the variation of the population-density difference induced by the modulation of the electric field is of order ϵ^2 while the amplitude of the electric field is of order ϵ. This fact is simply interpreted as follows: for an electric field of intensity *weak* compared with the size of the coupling constant, only a *small* proportion of the atoms are moved to the upper level (or excited) by the electric field. It is worth comparing this result with the low density case (3.5) where the variations of $\mathcal{E}$ and $\mathcal{N}$ are of *same order*, i.e. all the atoms are excited by the electric field. However, though being of order ϵ^2, the variation of $\mathcal{N}$ cannot be ignored as it is actually the very source of the nonlinearity, see (4.18).

5. Integrable limits

The system (4.14) has a simple integrable limit in the case of *circular polarization*, that is when

$$\phi = \pm\mathrm{i}\psi \tag{5.1}$$

for which it reduces to the scalar NLS equation

$$\mathrm{i}\lambda\partial_\tau\psi - \mu\partial_\xi^2\psi - 2\alpha|\psi|^2\psi = 0\,. \tag{5.2}$$

It is the so-called *focusing* nonlinear Schrödinger equation, possessing localized soliton solutions (see section 7). If the hypothesis (5.1) had been adopted right at the beginning of the study, one would have derived directly the NLS equation as the model for the interaction of a polarized electromagnetic field with a dense two-level medium.

Another integrable limit of our system would be obtained for $a = 1$ for which, at first sight, the system (4.14) reduces to the Manakov system [19]. However, setting $a = 1$ in (4.15) gives $\omega = 1$ and consequently $\lambda = \mu = 0$. Hence this value for a is forbidden in our model and the question arises as to whether or not the system has another integrable limit.

We consider the *random phase averaging limit*, obtained by setting

$$\psi = \psi' \, e^{i\theta_1} \qquad \phi = \phi' \, e^{i\theta_2} \tag{5.3}$$

where θ_1 and θ_2 depend on some external parameter, and by averaging (4.14) over the phase $\theta_1 - \theta_2$. In this case, the RHS of (4.14) vanishes and the equation becomes the Manakov system [19]

$$\begin{aligned} &i\lambda\partial_\tau\langle\psi'\rangle - \mu\partial_\xi^2\langle\psi'\rangle - \alpha\left(a|\langle\psi'\rangle|^2 + |\langle\phi'\rangle|^2\right)\langle\psi'\rangle = 0 \\ &i\lambda\partial_\tau\langle\phi'\rangle - \mu\partial_\xi^2\langle\phi'\rangle - \alpha\left(a|\langle\phi'\rangle|^2 + |\langle\psi'\rangle|^2\right)\langle\phi'\rangle = 0\,. \end{aligned} \tag{5.4}$$

This is actually a non-integrable system (except again for $a = 1$) [20].

Our system then appears to be more general than the Manakov equation and the question of its integrability will be considered through Painlevé analysis in section 7 . We will discuss in the following section some essential differences between the two models, based on the related respective conservation laws.

6. Conservation laws, Hamiltonian

It is possible to prove by direct calculations that the evolution (4.14) possesses three conserved flows, namely,

$$\partial_\tau(|\psi|^2 + |\phi|^2) = -i\mu\partial_\xi(\bar{\psi}_\xi\psi - \bar{\psi}\psi_\xi + \bar{\phi}_\xi\phi - \bar{\phi}\phi_\xi) \tag{6.1}$$

$$\partial_\tau(\bar{\psi}_\xi\psi - \bar{\psi}\psi_\xi + \bar{\phi}_\xi\phi - \bar{\phi}\phi_\xi) = -i\mu\partial_\xi(\bar{\psi}_{\xi\xi}\psi - \bar{\psi}\psi_{\xi\xi} + \bar{\phi}_{\xi\xi}\phi - \bar{\phi}\phi_{\xi\xi}) \tag{6.2}$$

$$\partial_\tau\mathcal{H} + \partial_\xi\mathcal{J} = 0\,. \tag{6.3}$$

In the last equation hereabove, the current $\mathcal{J}$ is given by

$$\begin{aligned} \mathcal{J} = \frac{i\mu}{2\lambda}\big[&-\mu\big(\bar{\psi}_{\xi\xi}\psi_\xi - \bar{\psi}_\xi\psi_{\xi\xi} + \bar{\phi}_{\xi\xi}\phi_\xi - \bar{\phi}_\xi\phi_{\xi\xi}\big) + 2\alpha\big(a|\psi|^2 + |\phi|^2\big) \\ &\times(\bar{\psi}_\xi\psi - \bar{\psi}\psi_\xi + \bar{\phi}_\xi\phi - \bar{\phi}\phi_\xi) - 2\alpha(a-1)\big(\phi_\xi\phi\bar{\psi}^2 - \bar{\psi}_\xi\bar{\psi}\phi^2\big)\big] \end{aligned} \tag{6.4}$$

and the Hamiltonian density $\mathcal{H}$ by

$$\mathcal{H} = \frac{1}{2\lambda}2\mu(|\psi_\xi|^2 + |\phi_\xi|^2) - 2\alpha|\psi|^2|\phi|^2 - a\alpha(|\psi|^4 + |\phi|^4) - \alpha(a-1)(\bar{\psi}^2\phi^2 + \bar{\phi}^2\psi^2)\,. \tag{6.5}$$

Our system (4.14) is Hamiltonian with respect to the following Poisson bracket:

$$\{A, B\} = -i\int\sum_{j=1}^{2}\left(\frac{\partial A}{\partial\psi_j}\frac{\partial B}{\partial\psi_j^*} - \frac{\partial A}{\partial\psi_j^*}\frac{\partial B}{\partial\psi_j}\right) \tag{6.6}$$

where

$$j = 1 \qquad \psi_1 = \psi \tag{6.7}$$

$$j = 2 \qquad \psi_2 = \phi\,. \tag{6.8}$$

Indeed Hamilton's equations of motion are then precisely (4.14), that is

$$\psi_\tau = \{\psi, \mathcal{H}\} \tag{6.9}$$

$$= \frac{-\mathrm{i}}{2\lambda}[-2\alpha(a|\psi|^2 + |\phi|^2)\psi - 2\alpha(a-1)\phi^2\bar{\psi} - 2\mu\psi_{\xi\xi}] \tag{6.10}$$

$$\phi_\tau = \{\phi, \mathcal{H}\} \tag{6.11}$$

$$= \frac{-\mathrm{i}}{2\lambda}[-2\alpha(a|\phi|^2 + |\psi|^2)\phi - 2\alpha(a-1)\psi^2\bar{\phi} - 2\mu\phi_{\xi\xi}]\,. \tag{6.12}$$

There is an important property which creates the difference between our system and the Manakov equation (5.4): in our case, *only the sum* $\int |\psi|^2 + |\phi|^2$ is a conserved quantity (by integration of (6.1)), while both terms are independently conserved in the Manakov case (5.4). Here we have indeed

$$\partial_\tau \int |\psi|^2\,\mathrm{d}\xi = -\partial_\tau \int |\phi|^2\,\mathrm{d}\xi = \mathrm{i}\alpha(a-1)\int (\bar{\phi}^2\psi^2 - \phi^2\bar{\psi}^2)\,\mathrm{d}\xi \tag{6.13}$$

and consequently there is an effective coupling between the two directions of polarization of the electric field. However, the system being not integrable, we can think of studying this coupling only in numerical experiments, which again will be the subject of further studies.

7. Painlevé analysis of the system

The purpose of the generalized Painlevé test on *non-integrability* is to determine whether the general solution of a partial differential equation (PDE) has critical points (e.g. singularities which are not poles) [24–27]. In such a case, the PDE does not pass the Painlevé test and is conjectured to be non-integrable.

A differential equation has the Painlevé property if its general solution u can be expanded locally in a Laurent-like series

$$u = \varphi^\alpha \sum_{j=0}^{\infty} u_j \varphi^j \tag{7.1}$$

where $u_0 \neq 0$, $\varphi = \varphi(\xi, \tau)$, $u_j = u_j(\xi, \tau)$ are analytic functions of (ξ, τ) in the neighbourhood of the singularity manifold

$$M = \{(\xi, \tau) : \varphi(\xi, \tau) = 0)\} \tag{7.2}$$

and where α is a negative integer.

The test consists of three steps:

(i) Determination of the leading-order behaviour α.
(ii) Search of the resonances which are the values of j, in the expansion (7.1), at which the corresponding u_j is an arbitrary function. Indeed, when it fails to be arbitrary, terms of the form $\varphi^j \ln\varphi$ must be included in the expansion, and this makes the solution multi-valued about the singularity manifold.
(iii) Verification that there exists a sufficient number of arbitrary functions.

Here we shall obtain the result that the *only* case in which the system (4.14):

$$\begin{aligned} &\mathrm{i}\partial_\tau\psi - \partial_\xi^2\psi - \left(a|\psi|^2 + |\phi|^2\right)\psi = (a-1)\phi^2\bar{\psi} \\ &\mathrm{i}\partial_\tau\phi - \partial_\xi^2\phi - \left(a|\phi|^2 + |\psi|^2\right)\phi = (a-1)\psi^2\bar{\phi}\,. \end{aligned} \tag{7.3}$$

(μ/α has been scaled off into ξ and λ/α into τ) passes the test corresponds to the value $a = 1$. However, as discussed in section 5, this case is forbidden and hence the system (4.14) *is not integrable.*

It is convenient first to take the complex conjugate of (4.14) and to consider the system of four coupled equations (where ψ, $\bar{\psi}$, ϕ and $\bar{\phi}$ are the independant variables). In order to find the leading order, let us set

$$\psi = x\,\varphi^{\alpha_1} \qquad \phi = y\,\varphi^{\alpha_2} \qquad \bar{\psi} = z\,\varphi^{\alpha_3} \qquad \bar{\phi} = w\,\varphi^{\alpha^4}\,. \tag{7.4}$$

Then, equating the dominanting terms, we obtain on the one hand

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -1 \tag{7.5}$$

and on the other hand the relations

$$\begin{aligned}
&(yz)^2 - (xw)^2 = 0 \\
&(x^2+z^2)[2\varphi_\xi^2 + axz + yw] + (a-1)xz(y^2+w^2) = 0 \\
&(y^2+w^2)[2\varphi_\xi^2 + ayw + xz] + (a-1)yw(x^2+z^2) = 0\,.
\end{aligned} \tag{7.6}$$

The solutions of (7.6) are

$$\begin{array}{lll}
x = 0 & y = -2\varphi_\xi^2/(aw) & z = 0 \\
x = 0 & y = \pm \mathrm{i}w & z = 0 \\
x = \pm\mathrm{i}\varphi_\xi^2/w & y = -\varphi_\xi^2/w & z = \pm\mathrm{i}w \\
x = \pm w & y = \pm\mathrm{i}w & z = \pm\mathrm{i}w \\
x = \pm w & y = \pm\mathrm{i}w & z = \mp\mathrm{i}w \\
x = \pm w & y = \mp\mathrm{i}w & z = \pm\mathrm{i}w \\
x = \pm w & y = \mp\mathrm{i}w & z = \mp\mathrm{i}w \\
x = \pm\mathrm{i}z & y = \pm\mathrm{i}w & \\
x = \pm\mathrm{i}z & y = \mp\mathrm{i}w\,. &
\end{array} \tag{7.7}$$

Note: the signs above are in one-to-one correspondance.

The next step consists in finding the resonances. One always occurs at j=-1 and corresponds to the arbitrariness of φ itself. The technique now consists in inserting the ansatz

$$\begin{aligned}
\psi &= x/\varphi + a_n\varphi^{n-1} \qquad & \phi &= y/\varphi + b_n\varphi^{n-1} \\
\bar{\psi} &= z/\varphi + c_n\varphi^{n-1} \qquad & \bar{\phi} &= w/\varphi + d_n\varphi^{n-1}
\end{aligned} \tag{7.8}$$

in the system (7.3). By comparison of the lowest-order terms we obtain the linear system

$$A.X_n = 0 \tag{7.9}$$

where

$$A = \begin{pmatrix}
N + 2axz + yw & ax^2 + (a-1)y^2 & xw + (a-1)2yz & xy \\
az^2 + (a-1)w^2 & N + 2axz + yw & wz & yz + (a-1)2wx \\
yz + (a-1)2xw & xy & N + 2ayw + xz & ay^2 + (a-1)x^2 \\
zw & xw + (a-1)2yz & aw^2 + (a-1)z^2 & N + 2ayw + xz
\end{pmatrix}$$

with $N = (n-1)(n-2)\varphi_\xi^2$ and

$$X_n = \begin{pmatrix} a_n \\ c_n \\ b_n \\ d_n \end{pmatrix}. \tag{7.10}$$

A non-trivial solution X_n requires $\det\{A\} = 0$. Among each of the14 different solutions (x, y, z or w cannot be taken as zero) of (7.7) we select those for which the resonances, e.g. the eight values of n, can just be integers. We find three of such solutions, namely

$$\begin{array}{lll} x = -\mathrm{i}\varphi_\xi^2/w & y = -\varphi_\xi^2/w & z = -\mathrm{i}w \\ x = \mathrm{i}\varphi_\xi^2/w & y = -\varphi_\xi^2/w & z = \mathrm{i}w \\ x = -\mathrm{i}\varphi_\xi^2/w & y = -\varphi_\xi^2/w & z = -\mathrm{i}w \end{array} \tag{7.11}$$

possessing the following resonances:

$$n = -1, 0, 3, 4, (3 \pm (-7 + 16a)^{1/2})/2, (3 \pm (-7 + 16a)^{1/2})/2\,. \tag{7.12}$$

As n has to be an integer, the quantity $(-7+16a)^{1/2}$ must be odd. Let consider the different cases. In the first one $(-7 + 16a = 1)$, the value of a would imply $3 + \omega^2 = 1 + \omega^2$ through (4.15), and this value must be dropped. In the second one $(-7 + 16a = 9)$, we obtain the value $a = 1$ and the system becomes then the Manakov system [19]. However, as we have seen in section 5, this value of a is forbidden in the physical situation considered. One can check easily that all other cases lead to a negative value for ω^2.

8. Stability analysis

Since the work of Benjamin and Feir [21] and later of Stuart and DiPrima [22], we know that the scalar (one-field) nonlinear Schrödinger equation governs two different regimes depending on the relative signs of the dispersive term versus the nonlinear term. In the defocusing case (opposite signs) the system is stable against small perturbations of the amplitude of the plane wave. It is said that it is *modulationally stable*. In the focusing case (same signs for the dispersive and nonlinear terms), it exists as a threshold for the wavenumber beyond which any perturbation of the envelope experiences an exponentional growth (modulational instability). This growth is rapidly saturated by the nonlinearity and thus the system propagates localized coherent structures: the solitons. The two regimes are thus physically very different and this is also true from a mathematical point of view. Indeed, in the spectral-transform scheme, the defocusing (stable) case is related to a self-adjoint eigenvalue problem (only real eigenvalues) while the focusing (unstable) case is related to a non-self-adjoint problem (complex eigenvalues related to the soliton solutions).

These stability properties are then crucial and we examine now our system (7.3) from the point of view of [21] and [22] and prove that it is actually *modulationally unstable*. As a consequence soliton solutions are naturally built up and we shall study these structures in the next section.

It is important for the following to remember that α, μ and a are *positive* constants. We look at the behaviour of a weak perturbation of the static solution

$$\psi_0(\tau) = \phi_0(\tau) = \exp\left[2\mathrm{i}\alpha\frac{a}{\lambda}\tau\right] \tag{8.1}$$

of the system (7.3) under the form

$$\psi = \psi_0(\tau)[1 + \tilde{\psi}(\xi, \tau)] \qquad \phi = \phi_0(\tau)[1 + \tilde{\phi}(\xi, \tau)]\,. \tag{8.2}$$

The perturbation is weak in the sense that all quadratic terms in (4.14) can be neglected and the resulting linear system reads

$$\begin{aligned} &-\mathrm{i}\lambda\partial_\tau\tilde{\psi} + \mu\partial_\xi^2\tilde{\psi} + \alpha[a(\tilde{\psi}^* + \tilde{\psi}) + (\tilde{\phi}^* + \tilde{\phi})] = b(2\tilde{\phi} + \tilde{\psi}^* - \tilde{\psi}) \\ &-\mathrm{i}\lambda\partial_\tau\tilde{\phi} + \mu\partial_\xi^2\tilde{\phi} + \alpha[a(\tilde{\phi}^* + \tilde{\phi}) + (\tilde{\psi}^* + \tilde{\psi})] = b(2\tilde{\psi} + \tilde{\phi}^* - \tilde{\phi})\,. \end{aligned} \tag{8.3}$$

A solution is now saught under the form of a plane wave of real wavenumber l and growth rate p (which can be complex) as

$$\begin{aligned}\tilde{\psi} &= A_1 \exp[(\mathrm{i}l\xi + p\tau)] + A_2 \exp[(-\mathrm{i}l\xi + p^*\tau)] \\ \tilde{\phi} &= B_1 \exp[(\mathrm{i}l\xi + p\tau)] + B_2 \exp[(-\mathrm{i}l\xi + p^*\tau)]\end{aligned} \tag{8.4}$$

where A_1, A_2, B_1 and B_2 are complex constants.

By inserting (8.4) in (8.3) and using the linear independence of the exponentials, one finds that (8.3) becomes a linear homogeneous system for the unknowns A_1, A_2, B_1 and B_2. A non-trivial solution requires a vanishing determinant, i.e.

$$[\lambda^2 p^2 + \mu l^2(\mu l^2 - 4\alpha a)][\lambda^2 p^2 + \mu l^2(\mu l^2 - 4\alpha(1-a))] = 0\,. \tag{8.5}$$

Requiring the first term to vanish, we get the following eigenrelation for p in terms of l:

$$p^2 = -\frac{\mu l^2}{\lambda^2}\left[\mu l^2 - 2\alpha\frac{3+\omega^2}{1+\omega^2}\right]. \tag{8.6}$$

Therefore, when

$$l < \sqrt{\frac{2\alpha(3+\omega^2)}{\mu(1+\omega^2)}} \tag{8.7}$$

the growth rate p is real-valued and the perturbations $\tilde{\psi}$ and $\tilde{\phi}$ grow exponentially in τ. This is the Benjamin–Feir instability which is a source of soliton formation as was first demonstrated in the context of water waves [23].

If now the second term of (8.5) is set equal to zero, the determining relation for the growth rate p becomes

$$p^2 = \frac{\mu l^2}{\lambda^2}\left[2\alpha\frac{\omega^2-1}{\omega^2+1} - \mu l^2\right]. \tag{8.8}$$

Consequently if $\omega < 1$ the growth rate p is a pure complex number and the solution is stable. If, however, $\omega > 1$, p takes real values in the range

$$l < \sqrt{\frac{2\alpha(\omega^2-1)}{\mu(1+\omega^2)}} \tag{8.9}$$

which is included in the preceding condition (8.7).

In summary, the Benjamin–Feir instability takes in our case a new feature when the parameter ω entering (4.14) (actually, the input frequency of the radiation interacting with the two-level system) is *less* than 1 (i.e. when we stay on the acoustic branch). In that case indeed, the system is not *strictly* unstable as it possesses altogether a stable and an unstable solution. There is then an open problem: which of these solutions is selected *naturally* by the system? Or shall we see bifurcations from a stable solution to the unstable one? We cannot answer these questions now, especially as the soliton solutions (described in the next section) do not strongly depend on the values of ω.

9. Solitary wave solutions

The simplest solitary wave solution to (4.14) is obtained in the case of a circularly polarized electromagnetic wave and reads

$$\psi(\xi,\tau) = \frac{-2\gamma\exp[2\mathrm{i}(\rho\sqrt{\alpha/\mu}\xi + (\rho^2-\gamma^2)(2\alpha\ /\lambda)\tau)]}{\cosh[2\gamma\sqrt{\alpha/\mu}(\xi + 4\rho(\alpha/\lambda)\sqrt{\mu/\alpha}\tau - \xi_0)]} = \pm\mathrm{i}\phi(\xi,\tau)\,. \tag{9.1}$$

This travelling envelope solution is characterized by two real parameters, its velocity

$$v_s = -4\frac{\rho}{\lambda}\sqrt{\alpha\mu} \tag{9.2}$$

and its amplitude -2γ. It moves in a frame (ξ, τ) which is in translation at the group velocity V given in (4.11); then there is the possibility that such a nonlinear coherent structure be trapped in the medium. To see this, it is necessary to come back to the laboratory frame (Z, T). Defining

$$\tilde{\gamma} = \epsilon\gamma \qquad \tilde{\rho} = \epsilon\rho \tag{9.3}$$

and using (4.4) and (4.5), the physical fields can be written independently of the scaling factor ϵ (the small quantity is now the amplitude $\tilde{\gamma}$ of the light pulse); they read

$$\mathcal{E}(Z,T) = \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \frac{-2\tilde{\gamma}\exp[\mathrm{i}\theta(Z,T)]}{\cosh\zeta(Z,T)} \exp[\mathrm{i}(\omega T - kZ)] + \mathcal{O}(\epsilon^2) \tag{9.4}$$

$$\mathcal{P}(Z,T) = -\begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \frac{\alpha\mathcal{N}_0}{1-\omega^2} \frac{-2\tilde{\gamma}\exp[\mathrm{i}\theta(Z,T)]}{\cosh\zeta(Z,T)} \exp[\mathrm{i}(\omega T - kZ)] + \mathcal{O}(\epsilon^2) \tag{9.5}$$

$$\mathcal{N}(Z,T) = \mathcal{N}_0\left(1 - \alpha\frac{1+\omega^2}{(1-\omega^2)^2}\right) \frac{8\tilde{\gamma}^2}{\cosh^2\zeta(Z,T)} + \mathcal{O}(\epsilon^3)\,. \tag{9.6}$$

Above we have defined

$$\theta(Z,T) = 2\tilde{\rho}\sqrt{\frac{\alpha}{\mu}}(Z - VT) + 4(\tilde{\rho}^2 - \tilde{\gamma}^2)\frac{\alpha}{\lambda}T \tag{9.7}$$

$$\zeta(Z,T) = 2\tilde{\gamma}\sqrt{\frac{\alpha}{\mu}}\left[(Z - VT) + 4\frac{\tilde{\rho}}{\lambda}\sqrt{\mu\alpha}T - Z_0)\right]. \tag{9.8}$$

As it should be, the above solution is independent of the scaling parameter ϵ, which confirms the consistency of the multiscale expansion of section 4.

Appendix.

Inserting the infinite series expansion (4.1) into the system (2.7) we obtain

$$\begin{aligned}
&[(\epsilon^2\partial_\tau - \epsilon V\partial_\xi + \mathrm{i}n\omega)^2 + 1](P_0^n + \epsilon P_1^n + \cdots) \\
&\qquad = -\alpha\sum_{p+q=n}(N_0^p + \epsilon N_1^p + \cdots)(E_0^q + \epsilon E_1^q + \cdots)
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
&(\epsilon^2\partial_\tau - \epsilon V\partial_\xi + \mathrm{i}n\omega)(N_0^n + \epsilon N_1^n + \cdots) \\
&\qquad = \sum_{p+q=n}[(\epsilon^2\partial_\tau - \epsilon V\partial_\xi + \mathrm{i}q\omega)(P_0^q + \epsilon P_1^q + \cdots)](E_0^p + \epsilon E_1^p + \cdots)
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
&[(\epsilon^2\partial_\tau - \epsilon V\partial_\xi + \mathrm{i}n\omega)^2 - (\epsilon\partial_\xi - \mathrm{i}nk)^2](E_{0x}^n + \epsilon E_{1x}^n + \cdots) \\
&\qquad = -(\epsilon^2\partial_\tau - \epsilon V\partial_\xi + \mathrm{i}n\omega)^2(P_{0x}^n + \epsilon P_{1x}^n + \cdots)
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
&[(\epsilon^2\partial_\tau - \epsilon V\partial_\xi + \mathrm{i}n\omega)^2](E_{0z}^n + \epsilon E_{1z}^n + \ldots) \\
&\qquad = -(\epsilon^2\partial_\tau - \epsilon V\partial_\xi + \mathrm{i}n\omega)^2(P_{0z}^n + \epsilon P_{1z}^n + \cdots)\,.
\end{aligned} \tag{A.4}$$

Hereabove, the quantities E_j^n and P_j^n are vectors of $\mathbb{C}^3$ and the indices x, y or z indicate their three components. Wherever the x-component appears, it is understood by symmetry that the same equation also holds for the y-component.

We now consider the above equations at all orders, and determine the series of constraints which leads to a closed non-trivial limit system.

Order $j = 0$. The hypothesis (4.3) ensures that the order 0 is automatically verified for any N_0^0 (the initial population difference between the two levels).

Order $j = 1$. For $n = 0$ we obtain

$$P_1^0 = -\alpha N_0^0 E_1^0 \tag{A.5}$$

and for $n = 1$:

$$\begin{aligned} &(1-\omega^2)P_1^1 = -\alpha N_0^0 E_1^1 \qquad && \mathrm{i}\omega N_1^1 = 0 \\ &(k^2-\omega^2)E_{1x}^1 = \omega^2 P_{1x}^1 \qquad && -\omega^2 E_{1z}^1 = \omega^2 P_{1z}^1 . \end{aligned} \tag{A.6}$$

Therefore comparing the third component of the above equations we obtain

$$E_{1z}^1 = P_{1z}^1 = 0 \tag{A.7}$$

and then the remaining equations have a non-trivial solution if and only if

$$\det \begin{pmatrix} 1-\omega^2 & \alpha N_0^0 \\ \omega^2 & \omega^2 - k^2 \end{pmatrix} = 0 \tag{A.8}$$

which is precisely the dispersion relation (2.14).

For $n > 1$ using the dispersion relation, it is easy to prove that

$$E_{1z}^n = P_{1z}^n = E_{1x}^n = P_{1x}^n = 0 \tag{A.9}$$

and hence the relation (4.6) is proved.

Order $j = 2$. For $n = 0$ we have

$$P_2^0 = -\alpha(N_0^0 E_2^0 + N_1^0 E_1^0) \tag{A.10}$$

$$-V\partial_\xi N_1^0 = 0 \tag{A.11}$$

and for $n = 1$:

$$(1-\omega^2)P_2^1 - 2\mathrm{i}\omega V \partial_\xi P_1^1 = -\alpha(N_0^0 E_2^1 + N_1^0 E_1^1) \tag{A.12}$$

$$\mathrm{i}\omega N_2^1 = \mathrm{i}\omega P_1^1 E_1^0 \tag{A.13}$$

$$(k^2-\omega^2)E_{2x}^1 - 2\mathrm{i}\omega V\partial_\xi E_{1x}^1 + 2\mathrm{i}k\partial_\xi E_{1x}^1 = \omega^2 P_{2x}^1 + 2\mathrm{i}\omega V \partial_\xi P_{1x}^1 \tag{A.14}$$

$$-\omega^2 E_{2z}^1 = \omega^2 P_{2z}^1 . \tag{A.15}$$

Consequently the fields are polarized also at the order ϵ^2:

$$E_{2z}^1 = P_{2z}^1 = 0 . \tag{A.16}$$

Using the first equation in x and comparing with the third one, we obtain a system of two equations where, thanks to the dispersion relation we can eliminate the terms E_{2x}^1 and P_{2x}^1. Hence we deduce

$$A\partial_\xi E_{1x}^1 = \alpha\omega^2 N_1^0 E_{1x}^1 \tag{A.17}$$

where the constant A is given by

$$\frac{1}{2\mathrm{i}} A = (1-\omega^2)(\omega V - k) - \omega V \alpha N_0^0 \frac{1}{1-\omega^2} . \tag{A.18}$$

It is now remarkable that, if the translation velocity V equals the group velocity according to (4.11), then $A = 0$ and hence (4.12) is proved. If, however, we chose an arbitrary velocity V, then the system would not close.

For $n > 1$, thanks again to the dispersion relation we have

$$E_2^n = P_2^n = 0 \qquad \text{for } n > 1 \tag{A.19}$$

$$2\mathrm{i}\omega N_2^2 = \mathrm{i}\omega P_1^1 E_1^1 \tag{A.20}$$

$$N_2^n = 0 \qquad \text{for } n > 2 . \tag{A.21}$$

Order $j = 3$. For $n = 0$ the system (2.7) gives

$$P_3^0 + V^2\partial_\xi^2 P_1^0 = -\alpha(N_0^0 E_3^0 + N_2^0 E_1^0 + N_2^1 E_1^{1*} + N_2^{1*} E_1^1) \tag{A.22}$$

$$V\partial_\xi(P_1^1 E_1^{1*} - P_1^{1*} E_1^1 - N_2^0) + V(\partial_\xi P_1^0)E_1^0$$
$$= \mathrm{i}\omega(P_1^1 E_2^{1*} - P_1^{1*} E_2^1 + P_2^1 E_1^{1*} - P_2^{1*} E_1^1)V(\partial_\xi P_1^0)E_1^0 \tag{A.23}$$

$$(V^2 - 1)\partial_\xi^2 E_{1x}^0 = -V^2\partial_\xi^2 P_{1x}^0 \tag{A.24}$$

$$V^2\partial_\xi^2 E_{1z}^0 = -V^2\partial_\xi^2 P_{1z}^0 . \tag{A.25}$$

Consequently, from the relations (A.5) and (4.3) we obtain in particular

$$N_2^1 = 0 . \tag{A.26}$$

For $n = 1$, the relevant equations read

$$(1 - \omega^2)P_3^1 - 2\mathrm{i}\omega V\partial_\xi P_2^1 + 2\mathrm{i}\omega\partial_\tau P_1^1 + V^2\partial_\xi^2 P_1^1 = -\alpha(N_0^0 E_3^1 + N_2^0 E_1^1 + N_2^2 E_1^{1*}) \tag{A.27}$$

$$(k^2 - \omega^2)E_{3x}^1 - 2\mathrm{i}\omega V\partial_\xi E_{2x}^1 + 2\mathrm{i}\omega\partial_\tau E_{1x}^1 + V^2\partial_\xi^2 E_{1x}^1 + 2\mathrm{i}k\partial_\xi E_{2x}^1 - \partial_\xi^2 E_{1x}^1$$
$$-V^2\partial_\xi^2 P_{1x}^1 + \omega^2 P_{3x}^1 - 2\mathrm{i}\omega\partial_\tau P_{1x}^1 + 2\mathrm{i}\omega V\partial_\xi P_{2x}^1 . \tag{A.28}$$

The quantities P_3^1 and E_3^1 can be eliminated from the above set of equations and, using (A.6), P_1^1 is expressed in terms of E_1^1, which finally produce a system for the only field E_1^1 (note that N_2^2 is also expressed in terms of E_1^1 by means of (A.20)). This system can be finally put into the form (4.14).

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